THE STRUCTURE OF SOLUTIONS TO PLATEAU'S PROBLEM IN THE n-SPHERE

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1. Introduction

Let T be a k-dimensional rectifiable current in the unit sphere S^n which is absolutely area minimizing with respect to S^n and is such that ∂T lies in a *closed* m-dimensional geodesic hemisphere Q. We will present results concerning the location of T (Theorems 3.5 and 4.4) and, in case $k = m$, the structure of T (Theorem 4.5). The primary difficulty arises from the assumption that Q is closed; simple examples using lines of longitude on S^2 show that not only is T not uniquely determined by ∂T, but there may be a continum of solutions to the Plateau problem for a fixed boundary lying in Q.

Our results are relevant to the study of the structure of oriented tangent cones at points on the boundary of an area minimizing current in R^n (see [3, 5.2]), and this was our principal motivation for undertaking this study. An application to this problem is given in § 5.

In order to obtain our main results we first prove a "location theorem" for minimizing and minimal (or stationary) currents of arbitrary dimension in S^n which is a formulation for currents in the sphere of the classical idea that a bounded minimal submanifold of R^n must lie in the convex hull of its boundary (a simple proof of which is also given). Such results were first obtained by Blaine Lawson [7] for smooth minimal immersions of manifolds of arbitrary dimension, and for pseudo-immersions in the two dimensional case. We will use his formulation of the notion of convex hull of a subset of S^n. The minimal "surfaces" which we consider include Lawson's as special cases; however, because his proofs are centered around use of a maximal principle, his results are stronger than ours.

The author is indebted to his colleague Benjamin Halpern for several stimulating discussions which lead to the use of the function F in the proof of Sheeting lemma 4.4 and Theorem 4.5. This construction has also been recently applied by Sandra Paur in her study of boundary behavior of integral currents [8].

2. Preliminaries

The purpose of this section is to fix basic notation and terminology, and discuss general concepts which will be used in the paper. Notations which are not explained below may be found in [4, pp. 669–671] and [1].

2.1. Throughout the paper, k and n will be integers with $1 \leq k \leq n$. Denote by R^n the n-dimensional Euclidean space with the standard inner product

$$x \cdot y = x^1 y^1 + \cdots + x^n y^n \qquad \text{for } x, y \in R^n ,$$

standard orthonormal basis $e_1, \cdots, e_n$ and norm

$$|x| = (x \cdot x)^{1/2} \qquad \text{for } x \in R^n .$$

Denote $S^n = R^{n+1} \cap \{x : |x| = 1\}$.

Whenever X is a set, $\mathbf{1}_X$ denotes the identity map of X.

Whenever M is a differentiable manifold and $x \in M$, one denotes the *tangent space of M at x* by $T_x(M)$.

2.2. Currents. Let M be a Riemannian manifold of class ∞. $\mathscr{R}_k(M)$ is the group of k-dimensional *rectifiable* currents with (compact) support in M. $\mathscr{R}_k^{\mathrm{loc}}(M)$ is the group of *locally rectifiable* currents in M.

Let N be a proper submanifold of class 1 of M. The second corollary on [4, p. 373] implies that

$$\mathscr{R}_k(N) = \mathscr{R}_k(M) \cap \{T : \operatorname{spt} T \subset N\} .$$

If T is a k-dimensional current in M, then $M(T)$ is the *mass* of T. In case $T \in \mathscr{R}_k^{\mathrm{loc}}(M)$, the *variation measure* $\|T\|$ of T is defined and one has the representation

$$T(\varphi) = \int \langle \vec{T}, \varphi \rangle \, d \, \|T\|$$

whenever φ is a continuous differential k-form with compact support in M, where $\vec{T}(x)$ is a unit simple k-vector for $\|T\|$ almost all $x \in M$.

2.3. Varifolds. Let $G(n, k)$ denote the *Grassmann manifold of k-dimensional linear subspaces of R^n*. Suppose $S \in G(n, k)$. We will also use S to denote orthogonal projection of R^n on S. With M as in § 2.2 we denote by $G_k(M)$ *the bundle of k-dimensional linear subspaces of tangent spaces of M*.

A *k-dimensional varifold in M* is a Radon measure V on $G_k(M)$. $V_k(M)$ is the weakly topologized space of k-dimensional varifolds in M. Whenever $V \in V_k(M)$, let

$$\|V\| (A) = V(G_k(M) \cap \{(x, S) : x \in A, \, S \subset T_x(M)\}) \qquad \text{for } A \subset M .$$

Clearly $\|V\|$ is a Radon measure on M.

Whenever $V \in V_k(M)$ and X is a Borel subset of M, let

$$V_X = V \, \lfloor \, (G_k(M) \cap \{(x, S) : x \in X, \, S \in T_k(M)\}) \; .$$

Clearly $V_X \in V_k(M)$ and $\|V_X\| = \|V\| \, \lfloor \, X$.

Finally, we observe that $T \in \mathscr{R}_k^{\mathrm{loc}}(M)$ can be identified with a varifold, also denoted T, whose value on a subset B of $G_k(M)$ is

$$\|T\| \, \{x : (x, S) \in B \text{ and } S \text{ is associated with } \bar{T}(x)\} \; .$$

2.4.   Minimizing currents.   Let A be a subset of R^n. A current $T \in \mathscr{R}_k(R^n)$ is called (*absolutely*) *area minimizing with respect to* A if spt $T \subset A$ and

$$M(T) \leq M(T + X)$$

whenever $X \in \mathscr{R}_k(R^n)$, spt $X \subset A$, $\partial X = 0$.

In case A is a compact Lipschitz neighborhood retract, the existence of certain minimizing currents was obtained in [4, 5.1.6].

These concepts are extended to the case where S is *locally rectifiable* by requiring that $T \, \lfloor \, K$ be absolutely area minimizing with respect to A for all compact subsets K of A.

2.5.   First variation.   Let M be a proper submanifold of R^n of class ∞. One denotes by $\mathscr{X}(M)$ the vector space of functions $g : M \to R^n$ of class ∞ with compact support such that $g(x) \in T_x(M)$ for $x \in M$.

Assuming $V \in V_k(M)$ we define a continuous linear function

$$\delta V : \mathscr{X}(M) \to R \; ,$$

called the *first variation* of V, by letting

$$\delta V(g) = \int_{G_k(M)} \text{trace} \, [S \circ Dg(x) \circ S^*] \, dV(x, S) \; .$$

Here, for $S \in G(n, k)$, S^* denotes the adjoint of S, that is, the inclusion of this subspace in R^n. (See [1, 4.2].)

One says that V is *stationary* if $\delta V = 0$. V is *stationary in* U if U is an open subset of M and $\delta V(g) = 0$ whenever $g \in \mathscr{X}(U)$.

This definition is motivated by the following (compare [1, 4.1]): Suppose $\varepsilon > 0$, $I = (-\varepsilon, \varepsilon)$, U is an open subset of M,

$$h : I \times M \to M \text{ is of class } \infty \; ,$$
$$h_t(x) = h(t, x) \text{ for } (t, x) \in I \times M, \, h_0(x) = x \text{ for } x \in M \; ,$$

and

$$\{x : h_t(x) \neq x \text{ for some } t \in I\} \text{ has compact closure in } U \; .$$

Let $g(x) = dh_t(x)/dt|_{t=0}$. If $\|V\|(U) < \infty$, then

$$\frac{d}{dt}\|h_{t\#}V\|(U)|_{t=0} = \delta V(g) \ .$$

The left side of this equation is classically referred to as the first variation of V associated with the isotopic deformation h.

If $T \in \mathscr{R}_k^{loc}(M)$ is area minimizing with respect to M, then T is stationary in $M \sim \mathrm{spt}\ \partial T$. See [4, p. 525].

Finally, we remark that if $T = i_\# N$ where N is an oriented manifold with boundary and $i: N \to M$ is a minimal immersion of class ∞ (in the sense of differential geometry), then T is stationary in $M \sim f\ (\mathrm{spt}\ \partial N)$; see [1, 4.2]. Furthermore, it follows from the result mentioned in the preceding paragraph that in case $k = 2$ and i is a minimal pseudo-immersion in the sense of Lawson [7, p. 226], T is stationary in $M \sim f\ (\mathrm{spt}\ \partial N)$.

2.6.   Assuming $C \in \mathscr{R}_{k+1}^{loc}(R^{n+1})$ we recall that C is an *oriented cone* if and only if

$$\boldsymbol{u}_{r\#}C = C \qquad \text{whenever } r > 1 \ ,$$

where $\boldsymbol{u}_r(x) = rx$ for $x \in R^{n+1}$.

Denoting

$$r(x) = |x| \qquad \text{for } x \in R^{n+1} \ ,$$
$$h(t, x) = tx \qquad \text{for } (t, x) \in R \times R^{n+1} \ ,$$

we also recall from [4, p. 452] and 2.2 that

$$C \cap S^n = \langle C, r, 1 \rangle \in \mathscr{R}_k(S^n) \ , \qquad C = h_\#[(E^1 \, \llcorner \, R^+) \times (C \cap S^n)] \ .$$

Finally, we recall from [4, p. 454] that $x \wedge \bar{C}(x) = 0$ for $\|C\|$ almost all x and use [4, 4.3.8] to conclude that

$$\bar{C}(x) = x \wedge (C \cap S^n)^-(x/|x|) \qquad \text{for } \|C\| \text{ almost all } x \ .$$

Lemma.   *Let $C \in \mathscr{R}_{k+1}^{loc}(R^{n+1})$ be an oriented cone such that $C \cap S^n$ minimizes area with respect to S^n. Then C minimizes area.*

Proof.   Suppose $s > 0$ and $T \in \mathscr{R}_{k+1}(R^{n+1})$ with $\partial T = 0$. Then for $\mathscr{L}^1$ almost all $t > 0$, $\partial\langle T, r, t \rangle = -\langle \partial T, r, t \rangle = 0$, whence using [4, 4.3.2 (2)] we infer that

$$M[C \, \llcorner \, B(0, s) + T] \geq \int_0^s M\langle C \, \llcorner \, B(0, s) + T, r, t \rangle \, dt$$

$$\geq \int_0^s M\langle C, r, t \rangle \, d\mathscr{L}^1 t = M[C \, \llcorner \, B(0, s)] \ ,$$

and conclude from application of [4, 5.4.2] with $Q_i = C \, \lfloor \, B(0, i)$ that C minimizes area. (Here, $B(0, s) = R^{n+1} \cap \{x : |x| \leq s\}$.)

3. Location theorems

The purpose of this section is to obtain geometric bounds for minimizing and stationary currents in S^n entirely in terms of the boundary of the current. (Corresponding results are obtained for varifolds, where appropriate.) For currents in R^n the analogous problem is quite simple; however, the existence of compact (closed) minimal submanifolds of S^n complicates the situation for stationary currents in S^n. Major difficulties arise when the current is not contained in some open hemisphere of S^n. Indeed, even if the boundary of the current lies in a hemisphere, quite clearly the current itself may not. (See [7, Examples, pp. 228–229].)

3.1. Lemma. *Suppose $\alpha \in S^n$, $a > 0$ and $V \in V_k(S^n)$ with*

$$\mathrm{spt} \, \| V \| \subset R^{n+1} \cap \{x : x \cdot \alpha \geq 0\} \ .$$

If V is stationary in $S^n \cap \{x : x \cdot \alpha < a\}$, then

$$\mathrm{spt} \, \| V \| \subset R^{n+1} \cap \{x : x \cdot \alpha \geq a\} \cup \{x : x \cdot \alpha = 0\}) \ .$$

Proof. We can assume $\alpha = e_1$. Denote

$$g_0(x) = e_1 - x^1 x \qquad \text{for } x = (x^1, \cdots, x^{n+1}) \in R^{n+1} \ ,$$

and fix $\psi \in \mathcal{E}^0(R)$ with $\psi \geq 0$, $\psi' \leq 0$, $\mathrm{spt} \, \psi \subset R \cap \{t : t < a\}$. Denote

$$g(x) = \psi(x^1) g_0(x) \qquad \text{for } x \in R^{n+1} \ .$$

Then $g \, | \, S^n \in \mathscr{X}(S^n \cap \{x : x^1 < \alpha\})$, hence $\delta V(g) = 0$.

On the other hand, fixing x, $v \in S^{n-1}$ such that $v \cdot x = 0$ one computes

$$\langle v, Dg_0(x) \rangle \cdot v = -x^1 \ ,$$
$$\langle v, Dg(x) \rangle = \psi'(x^1) v^1 g_0(x) + \psi(x^1) \langle v, Dg_0(x) \rangle \ ,$$
$$\langle v, Dg(x) \rangle \cdot v \leq -\psi(x^1) x^1 \ ,$$
$$\mathrm{trace} \, [S \circ Dg(x) \circ S^*] \leq -k\psi(x^1) x^1$$

for $S \in G(n + 1, k)$ such that $S(x) = 0$, hence concludes that

$$\int \psi(x^1) x^1 \, d \, \| V \| \, x = 0 \ .$$

3.2. Corollary. *Suppose $T \in \mathscr{R}_k(S^n)$ absolutely minimizes area with respect to S^n and*

$$\text{spt } \partial T \subset \mathbf{R}^{n+1} \cap \{x \colon x \cdot \alpha \geq a\} \ .$$

Then

$$\text{spt } T \subset \mathbf{R}^{n+1} \cap \{x \colon x \cdot \alpha \geq a\} \ .$$

Proof.   Assuming $\alpha = e_1$ we denote

$$C_+ = S^n \cap \{x \colon x^1 \geq a\} \ , \qquad C_- = S^n \cap \{x \colon x^1 \leq -a\} \ ,$$
$$C = S^n \sim (C_+ \cup C_-) \ ,$$

and define $\sigma \colon S^n \to S^n$ so that

$$\sigma(x) = (|x^1|, x^2, \cdots, x^{n+1}) \qquad \text{for } x \in \mathbf{R}^{n+1} \ .$$

Set $T_0 = \sigma_* T$. Inasmuch as $\partial T_0 = \partial T$, we see that $M(T_0) = M(T)$ and thus infer from Lemma 3.1 the existence of T', $T'' \in \mathscr{R}_k(S^n)$ such that $T_0 = T' + T''$ and

$$\text{spt } T' \subset S^n \cap \{x \colon x^1 = 0\} \ , \qquad \text{spt } T'' \subset C_+ \ .$$

Thus $\partial T'' = \partial T$. Hence $T' = 0$, and we conclude that $\sigma_*(T \lfloor C) = 0$. It then follows that

$$M(T_0) \leq M[\sigma_*(T \lfloor C_-)] + M(\sigma_*(T \lfloor C_+))$$
$$= M(T \lfloor C_-) + M(T \lfloor C_+) \leq M(T) \ ,$$

which implies that $T \lfloor C = 0$. Hence $T \lfloor C_- = 0$ because spt $\partial T \subset C_+$.

3.3.   Remark.   The result for varifolds in $\mathbf{R}^n$ which corresponds to the lemma is proved in much the same way:

Suppose $\alpha \in S^{n-1}$, $a \in R$, $V \in V_k(\mathbf{R}^n)$, and spt $\|V\|$ is compact. If V is stationary in $\mathbf{R}^n \cap \{x \colon x \cdot \alpha < a\}$, then

$$\text{spt } \|V\| \subset \mathbf{R}^n \cap \{x \colon x \cdot \alpha \geq a\} \ .$$

For the proof one assumes $\alpha = e_1$ and chooses ψ as before. Then

$$0 = \delta V(\psi e_1) = \int_{\mathbf{R}^n \times G(n,k)} \psi'(x^1) \, |S(e_1)|^2 \, dV(x,S) \ ,$$

whence it follows that

$$V \lfloor [\mathbf{R}^n \cap \{x \colon x^1 < a\} \times G(n,k)]$$
$$= V \lfloor [\mathbf{R}^n \cap \{x \colon x^1 < a\} \times G(n,k) \cap \{S \colon S(e_1) = 0\}] \ .$$

One uses this to verify that

$$0 = \delta V(\psi 1_{R_n}) = k \int_{\mathbf{R}^n} \psi(x^1) \, d \, \|V\| \, x \ .$$

3.4. Convexity. Following the ideas of Lawson [7, § 3] we will discuss the concept of convex hull of a set in S^n. Let $\mathcal{H}$ denote the set of closed hemispheres of S^n. To each $H \in \mathcal{H}$ one corresponds the unique vector $a(H) \in S^n$ such that

$$H = S^n \cap \{x : a(H) \cdot x \geq 0\} .$$

a is clearly one-to-one onto S^n; one introduces on $\mathcal{H}$ the unique differentiable and Riemannian structures making a an isometry.

Whenever $X \subset S^n$ denote

$$\mathcal{H}_X = \mathcal{H} \cap \{H : X \subset H\} , \qquad \bar{\mathcal{H}}_X = \mathcal{H} \cap \{H : X \subset \text{int } H\} ,$$
$$X^* = S^n \cap \{\alpha : \alpha \cdot x \geq 0 \text{ for } x \in X\} .$$

Clearly $\mathcal{H}_X$ is closed. Moreover, the function φ_X on $\mathcal{H}$ defined by

$$\varphi_X(H) = \text{dist}(X, -H)$$

is continuous (where $-H = S^n \cap \{-x : x \in H\}$). Thus, when X is closed, $\bar{\mathcal{H}}_X = \mathcal{H} \cap \{H : \varphi_X(H) > 0\}$, and $\bar{\mathcal{H}}_X$ is open.

The *convex hull* of $X \subset S^n$ is

$$\mathcal{C}(X) = \cap \, \mathcal{H}_X .$$

One says that X is *geodesically convex* if, whenever $x, y \in X$ and $x \neq -y$, the shortest geodesic arc joining x and y lies in X.

We will apply the following propositions which are proved in [7, § 3]: Let $X \subset S^n$.

3.4.1. Proposition. $\bar{\mathcal{H}}_X$ *is pathwise connected.*

3.4.2. Proposition. $\mathcal{C}(X)$ *is the smallest closed, geodesically convex set containing* X.

3.4.3. Proposition. *If X is closed and $\bar{\mathcal{H}}_X \neq \varnothing$, then*

$$\mathcal{C}(X) = \cap \, \bar{\mathcal{H}}_X .$$

3.5. Theorem. *Suppose $T \in \mathcal{R}_k(S^n)$ absolutely minimizes area with respect to S^n and spt ∂T lies in an open hemisphere. Then*

$$\text{spt } T \subset \mathcal{C}(\text{spt } \partial T) .$$

Proof. Since $\bar{\mathcal{H}}_{\text{spt } \partial T} \neq \varnothing$, one has

$$\text{spt } T \subset \cap \, \bar{\mathcal{H}}_{\text{spt } \partial T} = \mathcal{C}(\text{spt } \partial T)$$

by Corollary 3.2 and Proposition 3.4.3.

3.6. Theorem. *Suppose $V \in V_k(S^n)$ is stationary in an open subset U of S^n with spt $\|V\| \subset$ closure U. If spt $\|V\|$ lies in an open hemisphere, then*

$$\mathrm{spt}\,\|V\| \subset \mathscr{C}[(\mathrm{spt}\,\|V\|) \cap (\mathrm{bdry}\,U)] \,.$$

Proof. The proof is essentially the same as that for [7, Theorem 1]. Denote $\Sigma = \mathrm{spt}\,\|V\|$, $\Gamma = \Sigma \cap (\mathrm{bdry}\,U)$. Suppose $\Sigma \subset \mathrm{int}\,H_0$, $H_0 \in \mathscr{H}$. We see from Lemma 3.1 that we can assume that Σ is connected and $\Gamma \neq \varnothing$. Fix $H_1 \in \mathscr{H}_\Gamma$, and choose a continuous function $\gamma : [0, 1] \to \mathscr{H}_\Gamma$ with $\gamma(t) = H_t$ for $0 \leq t \leq 1$. Then $\Phi = \varphi_\Sigma \circ \gamma$ is continuous on $[0, 1]$ and $\Phi(0) > 0$. Assuming $\Phi(1) = 0$ we set $\tau = \inf \Phi^{-1}\{0\}$; clearly $\Sigma \subset H_\tau$ and $\Sigma \cap (\mathrm{bdry}\,H_\tau) \neq \varnothing$. But this is impossible by Lemma 3.1 since Σ is connected and $\varnothing \neq \Gamma \subset \mathrm{int}\,H_\tau$. Thus $\Phi(1) > 0$, and using Proposition 3.4.1 we conclude that $\Sigma \subset \bigcap \mathscr{H}_\Gamma$.

3.7. Corollary. *Suppose* $T \in \mathscr{R}_k(S^n)$ *is stationary in* $S^n \sim \mathrm{spt}\,\partial T$, *and* $\mathrm{spt}\,T$ *lies in an open hemisphere. Then*

$$\mathrm{spt}\,T \subset \mathscr{C}\,(\mathrm{spt}\,\partial T) \,.$$

3.8. Remark. The propositions corresponding to Theorem 3.6 and Corollary 3.7 for varifolds and currents in R^n are immediate consequences of Remark 3.3:

Suppose $V \in V_k(R^n)$ *is stationary in an open set* U *with* $\mathrm{spt}\,\|V\|$ *compact and* $\mathrm{spt}\,\|V\| \subset \mathrm{closure}\,U$. *Then* $\mathrm{spt}\,\|V\|$ *lies in the convex hull of* $(\mathrm{spt}\,\|V\|) \cap (\mathrm{bdry}\,U)$.

Suppose $T \in \mathscr{R}_k(R^n)$ *is stationary in* $R^n \sim \mathrm{spt}\,\partial T$. *Then* $\mathrm{spt}\,T$ *lies in the convex hull of* $\mathrm{spt}\,\partial T$.

3.9. Corollary. *Suppose* $0 \neq V \in V_k(S^n)$ *is stationary, and let* S *be a totally geodesic embedding of* S^{n-1} *in* S^n.

(i) *Then* $S \cap \mathrm{spt}\,\|V\| \neq \varnothing$, *and* $\mathrm{spt}\,\|V\|$ *is not a subset of any open hemisphere of* S.

(ii) *Assume* $\mathrm{spt}\,\|V\| \sim S \neq \varnothing$, *and let* $\mathscr{D}$ *be a component of* $\mathrm{spt}\,\|V\| \sim S$. *Then* $\mathscr{B} = (\mathrm{closure}\,\mathscr{D}) \sim \mathscr{D}$ *is a closed subset of* S *which is not contained in any open hemisphere of* S.

Proof. (i) If $S \cap \mathrm{spt}\,\|V\| = \varnothing$, and D is a component of $\mathrm{spt}\,\|V\|$, then Lemma 3.1 applied to V_D implies that $D = \varnothing$.

(ii) Choose $H_0 \in \mathscr{H}$ so that $S = \mathrm{bdry}\,H_0$ and $\mathscr{D} \subset \mathrm{int}\,H_0$, and assume there exists $H_1 \in \mathscr{H}$ with $\mathscr{B} \subset \mathrm{int}\,H_1$. Clearly $\mathscr{B} \subset S$, and hence $\mathscr{B} = (\mathrm{closure}\,\mathscr{D}) \cap S$ is closed and there exists an open neighborhood W of $\mathscr{B}$ such that $\mathrm{closure}\,W \subset \mathrm{int}\,H_1$.

We now apply Theorem 3.6 with

$$U = S^n \sim (\mathscr{D} \cap \mathrm{bdry}\,W)$$

and V replaced by $V_{\mathscr{D} \sim W}$. Accordingly, inasmuch as closure $U = S^n$, $\mathrm{spt}\,\|V_{\mathscr{D} \sim W}\| = \mathscr{D} \sim W \subset \mathrm{int}\,H_0$ since $\mathscr{D} \sim W$ is closed, and $(\mathrm{spt}\,\|V_{\mathscr{D} \sim W}\|) \cap (\mathrm{bdry}\,U) = \mathscr{D} \cap \mathrm{bdry}\,W$, we infer using Proposition 3.4.3 that

$$\mathrm{spt}\,\|V_{\mathscr{D} \sim W}\| \subset \mathscr{C}(\mathscr{D} \cap \mathrm{bdry}\,W) \subset \mathrm{int}\,H_1 \,.$$

Thus spt $\|V_{\mathscr{D}}\| \subset$ spt $\|V_{\mathscr{D} \sim W}\| \cup$ (closure W) $\subset$ int H_1.

Next we apply Theorem 3.6 to $V_{\mathscr{D}}$ with $U = S^n \sim \mathscr{B}$ to infer that

$$\text{closure } \mathscr{D} = \text{spt} \|V_{\mathscr{D}}\| \subset \mathscr{C}(\mathscr{B}) \ .$$

Finally, $\mathscr{C}(\mathscr{B}) \subset S$ by Proposition 3.4.2, which is a contradiction.

4. Structure theorems

Identify R^{n+1} with $R^k \times R^{n-k+1}$, denote

$$P_0 = R^k \times \{0\} \subset R^{n+1} \ , \quad R^+ = R \cap \{r : r > 0\} \ , \quad Q_0 = R^k \times R^+ \ ,$$

orient P_0 and Q_0, and define

$$F : R^{n+1} \to R^k \times R \ , \qquad\qquad F(u, v) = (u, |v|) \ ,$$
$$\Phi : R^{n+1} \sim P_0 \to Q_0 \times S^{n-k} \ , \qquad \Phi(u, v) = (u, |v|, v/|v|) \ .$$

It is easy to see that $\Phi^{-1}(u, r, y) = (u, ry)$ for $(u, r, y) \in Q_0 \times S^{n-k}$, so that Φ is a diffeomorphism of class ∞ onto $Q_0 \times S^{n-k}$. Further, for $y \in S^{n-k}$ we denote

$$Q_y = R^{n+1} \cap \{(u, ry) : u \in R^k, r > 0\} = \Phi^{-1}(Q_0 \times \{y\}) \ .$$

Orient Q_y so that $\partial Q_y = P_0$.

4.1. Lemma. *$DF(z)$ is an orthogonal projection whenever $z \in R^{n+1} \sim P_0$.*

Proof. Denoting the standard orthonormal basis of R^k by $e_1, \cdots, e_k$ and setting $\Phi(z) = w = (u, r, y) \in Q_0 \times S^{n-k}$ we have $z \in Q_y$, $F(Q_y) = Q_0$,

$$D\Phi^{-1}(w)(e_i, 0, 0) = (e_1, 0) \ , \qquad i = 1, \cdots, k \ ,$$
$$D\Phi^{-1}(w)(0, 1, 0) = (0, y) \ ,$$
$$D\Phi^{-1}(w)(0, 0, w) = (0, rw) \qquad \text{for } w \in T_y(S^{n-k}) \ .$$

Accordingly, $D\Phi^{-1}(w) | T_{(u,r)}(Q_0) \oplus \{0\}$ is an isometry onto $T_z(Q_y)$, and $T_z(Q_y)$ is orthogonal to

$$D\Phi^{-1}(z)[\{0\} \oplus T_y(S^{n-k})] = \ker DF(z) \ .$$

4.2. Theorem. *Suppose $1 \leq l \leq k$, $T \in \mathscr{R}_l(S^n)$ absolutely minimizes area with respect to S^n, and P is a k-dimensional linear subspace of R^{n+1} such that spt $\partial T \subset P$. Then there exist $(k + 1)$-dimensional linear subspaces $L_1, \cdots, L_\nu$ containing P such that*

$$\text{spt } T \subset L_1 \cup \cdots \cup L_\nu \ .$$

Proof. We can assume $P = P_0$. Defining $\iota : R^k \times R \to R^{n+1} = R^k \times R \times R^{n-k}$

so that

$$\iota(u, r) = (u, r, 0) \qquad \text{for } (u, r) \in \mathbf{R}^k \times \mathbf{R} \,,$$

we denote $Q = \text{closure } \iota(Q_0)$. Set $f = \iota \circ F$. Denote by $C \in \mathscr{R}_{l+1}^{loc}(\mathbf{R}^n)$ the oriented cone such that $C \cap \mathbf{S}^n = T$ (see § 2.6.)

We will first show that for $\|C\|$ almost all $z \in \mathbf{R}^{n+1} \sim P$,

$$\vec{C}(z) \in \textstyle\bigwedge_{l+1} T_z(Q_y) \quad \text{where} \quad \Phi(z) = (x, y) \,.$$

Whenever $z \in \mathbf{S}^n \sim P$, $\vec{T}(z)$ is simple, and $|\vec{T}(z)| = 1$, we denote

$$h(z) = | \textstyle\bigwedge_l Df(z)[\vec{T}(z)]| \,.$$

Referring to § 2.6 and Lemma 4.1 we see that $h(z) \leq 1$, and equality is equivalent to our assertion concerning $\vec{T}(z)$ holding. In case $z \in P$, denote $h(z) = 1$. Since $f \mid \mathbf{R}^{n+1} \sim P$ is smooth, it is clear that

$$f_\# T(\omega) \leq \int_{\mathbf{R}^{n+1} \sim P} h \, d\|T\|$$

whenever $\omega \in \mathscr{D}^l(\mathbf{R}^{n+1} \sim P)$ with $M(\omega) \leq 1$, hence

$$\|f_\# T\| (\mathbf{R}^{n+1} \sim P) \leq \int_{\mathbf{R}^{n+1} \sim P} h \, d\|T\| \,.$$

Moreover, recalling § 2.2 we infer that $T \, \llcorner \, P$ and $(f_\# T) \, \llcorner \, P$ belong to $\mathscr{R}_l(P)$ and hence are equal. Consequently,

$$M(f_\# T) \leq \int h \, d\|T\| \,.$$

On the other hand, $f_\# T \in \mathscr{R}_l(\mathbf{S}^n)$ and $\partial f_\# T = \partial T$. Thus since T is area minimizing,

$$\int h \, d\|T\| \leq M(T) \leq M(f_\# T) \,,$$

and it follows that $h(z) = 1$ for $\|T\|$ almost all z.

Next consider $W_0 \in G(n + 1, 2)$ such that $W_0 \subset \{0\} \times \mathbf{R}^{n-k+1}$. Referring to [2, 5.1] we see that the measure T defined there is equal to zero, and hence $W_0 \cap (\text{spt } C) \cap \mathbf{S}^n$ is finite. Thus there exists a finite set $\mathscr{A}_0$ of $(n - k)$-dimensional linear subspaces of $\mathbf{R}^{n-k+1}$ such that

$$\text{spt } C \subset \bigcup \{P \times A : A \in \mathscr{A}_0\} \,.$$

Assuming $n - k \geq 2$ and fixing $A_0 \in \mathscr{A}_0$, we choose $W_1 \in G(n + 1, 2)$ such that $W_1 \subset A_0$ and infer the existence of $\mathscr{A}_1$ as before. Since $\dim A \cap A_0 = n - k - 1$ for $A \in \mathscr{A}_1$ and

$$(\operatorname{spt} C) \cap A_0 \subset \bigcup \{P \times (A \cap A_0) : A \in \mathscr{A}_1\} \,,$$

we infer the existence of a finite set $\mathscr{B}_1$ of $(n - k - 1)$-dimensional linear subspaces of R^{n-k+1} such that

$$\operatorname{spt} C \subset \bigcup \{P \times B : B \in \mathscr{B}_1\} \,.$$

Proceeding inductively we obtain for each $i = 1, \cdots, n - k - 1$ a finite set $\mathscr{B}_i$ of $(n - k - i)$-dimensional linear subspaces of R^{n-k+1} such that

$$\operatorname{spt} C \subset \bigcup \{P \times B : B \in \mathscr{B}_i\} ;$$

the requirements of our theorem are fulfilled by

$$\{P \times B : B \in \mathscr{B}_{n-k-1}\} \,.$$

4.3. Remark. In Theorem 4.5 we will give a complete description of the structure of T for the case where $l = k$. In case $l = k - 1$, one can use the regularity results of Almgren, Federer, Fleming and Simons (see [4, 5.4.15] and [5]) to obtain the following:

(i) *If $2 \leq k \leq 7$, then $\operatorname{spt} T \sim P$ is a proper $(k - 1)$-dimensional submanifold of class ∞ of S^n.*

(ii) *If $k \geq 8$, then there exists $Z \subset \operatorname{spt} T \sim P$ such that $\operatorname{spt} T \sim (Z \cup P)$ is a proper $(k - 1)$-dimensional submanifold of class ∞ of S^n and the Hausdorff dimension of Z is not greater than $k - 8$.*

4.4. Sheeting lemma. *Let M and N be Riemannian manifolds of class 1 with M connected and oriented, $\dim M = m$. Suppose $T \in \mathscr{R}_m^{\mathrm{loc}}(M \times N)$, $\partial T = 0$, and $\mathbf{M}(T \lfloor K \times N) < \infty$ whenever K is a compact subset of M. Further, assume that*

$$\vec{T}(x, y) \in \bigwedge{}_m T_x(M) \oplus \{0\} \qquad for \ \|T\| \ almost \ all \ (x, y) \in M \times N \,.$$

Then there exist $y_i \in N$ and integers α_i, $i = 1, \cdots, \nu$, such that

$$T = \sum_{i=1}^{\nu} \alpha_i M \times \{y_i\} \,.$$

Here $M \times \{y_i\}$ is also used to denote the current obtained by integration over $M \times \{y_i\}$.

Robert Hardt has recently devised a proof of this proposition, which is much more elegant than the proof we had planned for presentation here. We therefore omit our proof and refer the reader to [6].

4.5. Theorem. *Suppose $T \in \mathscr{R}_k(S^n)$ absolutely minimizes area with respect to S^n, and there exists a closed oriented $(k + 1)$-dimensional half-plane Q with $\operatorname{spt} \partial T \subset Q$. Denote $P = \partial Q$. Then there exist oriented $(k + 1)$-dimensional half-planes $Q_1, \cdots, Q_\nu$ and nonnegative integers $\alpha_1, \cdots, \alpha_\nu$ such that the*

following are true:

(i) $Q_j \neq Q$ *and* $\partial Q_j = P$ *for* $j = 1, \cdots, \nu$, *and*

$$T = \beta(\alpha_1 Q_1 + \cdots + \alpha_\nu Q_\nu) \cap S^n + T \, \lfloor \, Q, \ \beta = \pm 1 \ .$$

(ii) *If* spt $\partial T \subset P$, *then* $T \, \lfloor \, Q = \beta \alpha Q \cap S^n$ *where* α *is a nonnegative integer. Moreover,*

$$\partial T = \theta \partial (Q \cap S^n) = -\theta P \cap S^n \ , \qquad M(T) = \tfrac{1}{2}(k+1) a(k+1)\theta \ ,$$

where $\theta = \alpha + \alpha_1 + \cdots + \alpha_\nu$ *and* $\beta = \operatorname{sign} \theta$.

(iii) *If* $P \cap S^n \not\subset$ spt ∂T, *then* spt $T \subset Q$.

Proof. Define ι, f, C as in the proof of Theorem 4.2; we can assume $P = P_0$ and $Q = $ closure $\iota(Q_0)$. Note that $\Phi \circ \iota(Q_0) = Q_0 \times \{e_1\}$. Denote by $C_0 \in \mathscr{R}_{k+1}^{\mathrm{loc}}(R^{n+1} \sim Q)$ the restriction of C to $\mathscr{D}^{k+1}(R^{n+1} \sim Q)$. Then $\partial C_0 = 0$ and, recalling the first paragraph of the proof of Theorem 4.2, we can apply Sheeting lemma 4.4 with $M = Q_0$, $N = S^{n-k} \sim \{e_1\} = S^{n-k} \sim \Phi(Q)$ and T replaced by $\Phi_\# C_0$. Consequently, there exist $y_i \in S^{n-k} \sim \{e_1\}$ and integers $\alpha_i \neq 0$, $i = 1, \cdots, \nu$, such that

$$C_0 = \alpha_1 Q_1 + \cdots + \alpha_\nu Q_\nu \ , \qquad Q_i = Q_{y_i} \ ,$$

and using [4, 4.1.21] we conclude that

$$C = C \, \lfloor \, (R^{n+1} \sim P) = C \, \lfloor \, Q + \alpha_1 Q_1 + \cdots + \alpha_\nu Q_\nu \ .$$

Inasmuch as $T \, \lfloor \, (S^n \sim Q)$ minimizes area with respect to S^n, $C \, \lfloor \, (R^{n+1} \sim Q)$ minimizes area by § 2.6, and the methods of the last paragraph of the proof of [3, 5.1] can be used to show that either $\alpha_1, \cdots, \alpha_\nu$ are of the same sign or $\nu = 2$ and closure $(Q_1 \cup Q_2)$ is a linear subspace of R^{n+1}. But this cannot be the case because $T \, \lfloor \, (S^n \sim Q) = (\alpha_1 Q_1 + \alpha_2 Q_2) \cap S^n$ minimizes area with respect to S^n

This proves (i); the first part of (ii) follows upon replacing Q by P in the above argument. Since $T = C \cap S^n$, the formula for $M(T)$ is the last statement in [3, 5.2] (which also contains a different proof of our theorem for the case where spt $\partial T \subset P$).

With regard to (iii) we use the second corollary on [4, p. 373] and [4, 4.1.28] to obtain an $\mathscr{H}^k$ summable, integer valued function φ on $Q' = Q \cap S^n$ such that

$$T \, \lfloor \, Q = Q' \wedge \varphi \ .$$

Let U be a connected open subset of $S^n \sim$ spt ∂T such that $U \cap P \neq \varnothing$. Denote $\theta' = \alpha_1 + \cdots + \alpha_\nu$. Since $f_\# T = Q' \wedge (\varphi + \beta \theta')$ and $\partial f_\# T = f_\# \partial T$, the constancy theorem [4, p. 357] applied on

$$U \cap (R^k \times \{v : v^2 = \cdots = v^{n-k+1} = 0\})$$

(the intersection of U with the $(k + 1)$-plane containing Q) implies that $(\varphi + \beta\theta')|Q' \cap U$ may be taken to be zero. On the other hand,

$$\int_{Q'} |\varphi + \beta\theta'| \, d\mathcal{H}^k = M(f_* T) = M(T)$$

$$= M(T \llcorner Q') + \theta' \mathcal{H}^k(Q')$$

$$= \int_{Q'} |\varphi| + \theta' \, d\mathcal{H}^k .$$

Therefore, if θ' were positive, then we would have sign $\varphi(z) = \beta$ for $\mathcal{H}^k$ almost all $z \in Q'$ which is a contradiction.

5. Application to tangent cones

It is well-known (and not difficult to verify) that an oriented cone C is stationary in $R^n \sim \operatorname{spt} \partial C$ if and only if $C \cap S^{n-1}$ is stationary in $S^{n-1} \sim \operatorname{spt} \partial C$. The corresponding proposition for area minimizing cones is, however, not true (but recall § 2.6), and this is the major obstacle to the determination of the structure of those oriented tangent cones which occur at points on the boundary of an area minimizing current with smooth boundary. At present the only result known in this direction follows from [2, 5.1] (see also [3, 5.2]) and employs the additional assumption that $\operatorname{spt} C \sim \operatorname{spt} \partial C$ lies in an open half-space of R^n. In this section we will give, under a similar assumption, a result concerning the structure of certain oriented tangent cones which occur at "corners" of the boundary of an area minimizing current in R^n.

For an example of an area minimizing cone C such that $C \cap S^{n-1}$ does not minimize area, let $C = f_*(E^2 \llcorner A)$ where $f : R^2 \to R^n$ is one-to-one and linear, and

$$A = R^2 \cap \{(x, y) : x \le 0 \text{ or } y \le 0\} .$$

5.1. Theorem. *Let $C \in \mathcal{R}_k^{\mathrm{loc}}(R^n)$ be an oriented cone such that C is stationary in $R^n \sim \operatorname{spt} \partial C$. (This holds in particular if C absolutely minimizes area.) If there exists $\alpha \in S^{n-1}$ such that*

$$\operatorname{spt} C \sim \{0\} \subset R^n \cap \{x : x \cdot \alpha > 0\} ,$$

then $\operatorname{spt} C$ lies in the convex hull of $\operatorname{spt} \partial C$.

Proof. Apply Corollary 3.7 to $C \cap S^{n-1}$.

5.2. Corollary. *Assume in addition that $\operatorname{spt} \partial C$ is contained in a k-dimensional linear subspace L. Then $\operatorname{spt} C \subset L$, and C is the unique area minimizing oriented cone in R^n with boundary ∂C which lies in the half-space $R^n \cap \{x : x \cdot \alpha > 0\}$.*

Proof. $C \in \mathcal{R}_k^{\mathrm{loc}}(L)$ and is the unique member of $\mathcal{R}_k^{\mathrm{loc}}(L)$ with boundary ∂C which lies in the half-space $L \cap \{x : x \cdot \alpha > 0\}$.

5.3.   Let $\gamma \colon [a, b] \to R^n$ be a simple closed curve, and suppose $a = x_0 <$ $x_1 < \cdots < x_n = b$ are such that for each $i, \gamma \mid [x_i, x_{i-1}]$ has a nonvanishing derivative which satisfies a Hölder condition with exponent not greater than one. Fix i so that $0 < i < n$ and $\gamma'(x_i-) \neq \gamma'(x_i+)$.

5.4.   Corollary.   *Let $S \in I_2(R^n)$ be absolutely area minimizing and such that $\partial S = \theta \gamma_{\#}[a, b]$ where θ is a positive integer. Suppose there exist $\varepsilon > 0, \alpha \in S^{n-1}$ and a neighborhood U of $\gamma(x_i)$ such that*

$$(\mathrm{spt}\, S) \,\cap\, U \sim \{\gamma(x_i)\} \subset R^n \,\cap\, \{x \colon [x - \gamma(x_i)]\cdot\alpha \geq \varepsilon\} \,.$$

Then S has a unique oriented tangent cone C at $\gamma(x_i)$. Moreover, $\theta^{-1}C$ is the current represented by integration over the (suitably oriented) smaller of the sectors determined by $\gamma'(x_i-)$ and $\gamma'(x_i+)$ in the 2-dimensional plane spanned by these vectors.

Proof.   The existence of C follows from [3, 3.3, 3.6] and the proposition in [3, 2.6].

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